1. It is given that \( y_1 = x \) is a solution of \( (1-x^2)y'' - 2xy' + 2y = 0 \), \(-1 < x < 1\). Find the second linearly independent solution \( y_2 \) use reduction order method. 

(10%)  

2. (a) What is the smallest integer \( n > 0 \) for which there is a differential equation 
\[ y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 \] 
having among its solutions the function 
\[ y(t) = \cos 2t + 4t^2 e^t - e^{-t}. \]

(2%) 

(b) Find the constants \( a_1, a_2, \ldots, a_n \). 

(5%)  

3. Consider the linear ordinary differential equation \( \frac{dy}{dx} + p(x)y = f(x) \) 

(a) Prove that \( I(x) = \exp \left[ \int p(x)dx \right] \) is an integrating factor of the above equation. 

(5%) 

(b) If \( p(x) = 2x \), \( f(x) = 2x \cos(x^2) \), \( y(0) = 1 \), Solve \( y(x) \) 

(5%)  

4. Let \( x_1(t) \) and \( x_2(t) \) be two solution of the following differential equation: 
\[ \ddot{x}(t) - 3 \cos 2t \dot{x}(t) + 2e^{-2t} x(t) = 0 \] 
and their initial conditions satisfy the following equation: 
\[
\begin{align*}
x_1(0) + x_2(0) &= 0 \\
x_1(0) + x_2(0) + \dot{x}_1(0) &= 0 \\
x_1(0) + \dot{x}_2(0) + \dot{x}_2(0) &= 0 \\
\ddot{x}_1(0) + \ddot{x}_2(0) &= 0 \\
\dddot{x}_1(0) + \dddot{x}_2(0) &= 1
\end{align*}
\]

Are \( x_1(t) \) and \( x_2(t) \) linearly independent? Please prove your answer. 

(8%)
5. Find the Laplace transform of following given functions:
   
   (a) \[ f(t) = e^{-3t} \int_{0}^{\infty} \frac{\sin 2u}{u} \, du \]  
   (5%)

   (b) \[ f(t) = \begin{cases} 
   \frac{3}{2} t ; & t \in [0,2] \\
   0 ; & t \in [2,8] 
   \end{cases} \text{ and } f(t) = f(t+T) \]  
   (5%)

6. Let the linear mapping \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( L(x) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \) where \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).

   (a) Determine the matrix representation of \( L \) with respect to the standard basis.  
   (4%)

   (b) Repeat (a) with the basis changed to \( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).  
   (6%)

7. Given the matrix \( A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \), calculate \( A^4 \) and \( A^2 \).  
   (10%)

8. Find all real eigenvalues and corresponding eigenfunctions of the boundary value problem
   \[ y'' + \lambda y = 0, \quad 0 < x < \pi, \quad y(0) - y(\pi) = 0, \quad y'(0) - y'(\pi) = 0 \]  
   (10%)

9. (a) Find the Fourier cosine and Fourier sine integral of \( f(x) = xe^{-x}, x \geq 0 \).  
   (10%)

   (b) Find the Fourier transform of \( f(t) = e^{-3t}, -\infty < t < \infty \)  
   (5%)

10. (a) If \( f(z) = u(x,y) + iv(x,y) \) is analytic, prove that \( u(x,y) \) and \( v(x,y) \) are Harmonic Function.  
    (4%)

    (b) Evaluate the complex line integral \( \int_{C} \frac{1}{z^3(z+4)} \, dz \) using residue theorem.

    where \( C \) are: (1) \( |z| = 2 \) ; (2) \( |z+2| = 3 \)  
    (6%)